BERNSTEIN-SATO IDEALS AND LOCAL SYSTEMS

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ABSTRACT. The topology of smooth quasi-projective complex varieties is very restrictive. One aspect of this statement is the fact that natural strata of local systems, called cohomology jump loci, have a rigid structure: they are torsion-translated subtori in a complex torus. We propose and partially confirm a relation between Bernstein-Sato ideals and local systems. This relation gives yet a different point of view on the nature of the structure of cohomology jump loci of local systems. The main result is a partial generalization to the case of a collection of polynomials of the theorem of Malgrange and Kashiwara which states that the Bernstein-Sato polynomial of a hypersurface recovers the monodromy eigenvalues of the Milnor fibers of the hypersurface. We also address multi-variable versions of the Monodromy Conjecture and provide support for these in the case of hyperplane arrangements.

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1. INTRODUCTION

1.1. **Bernstein-Sato ideals and local systems.** We first propose a conjectural picture relating Bernstein-Sato ideals with local systems. It is known that the topology of smooth quasi-projective complex varieties is very restrictive. One aspect of this statement is the fact that natural strata of local systems, called cohomology jump loci, have a rigid structure: they are torsion-translated subtori in a complex torus, see Green - Lazarsfeld [15, 16], Arapura [1, 2], Simpson [34], Budur [7], Libgober [22], Dimca - Papadima - Suciu [14], Popa - Schnell [30], Dimca - Papadima

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[13]. The conjectural picture we propose gives yet a different point of view on the nature of the structure of cohomology jump loci of local systems.

To be more precise, let $F = (f_1, \ldots, f_r)$ be a collection of polynomials f_j in $\mathbb{C}[x_1, \ldots, x_n]$. The *Bernstein-Sato ideal of* F is the ideal B_F generated by polynomials $b \in \mathbb{C}[s_1, \ldots, s_r]$ such that

$$b(s_1, \ldots, s_r) f_1^{s_1} \cdots f_r^{s_r} = P f_1^{s_1+1} \cdots f_r^{s_r+1}$$

for some algebraic differential operator

$$P \in \mathbb{C}\left[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, s_1, \ldots, s_r\right].$$

The existence of non-zero Bernstein-Sato ideals B_F has been proved by Sabbah [31], see also Bahloul [3] and Gyoja [17]. In the one-variable case $r = 1$, the monic generator of the ideal B_F is the classical Bernstein-Sato polynomial.

We conjecture that the zero locus of the Bernstein-Sato ideal is a subspace arrangement of a certain arithmetic type.

Conjecture 1. *The Bernstein-Sato ideal B_F is generated by products of linear polynomials of the form*

$$\alpha_1 s_1 + \ldots + \alpha_r s_r + \alpha$$

with $\alpha_i \in \mathbb{N}$, $\gcd(\alpha_1, \ldots, \alpha_r) = 1$, and $\alpha \in \mathbb{Q}_{>0}$.

This refines a result of Sabbah [31] and Gyoja [17] which states that B_F contains at least one element of this type. In the one-variable case $r = 1$, this is due to Kashiwara [18]. When $n = 2$, every element of B_F is divisible by the linear polynomials defining $(r-1)$-dimensional faces of the jumping polytopes of the local mixed multiplier ideals of $f_1, \ldots, f_r$, by Cassou-Noguès and Libgober [9, Theorem 4.1].

A slightly weaker version of Conjecture 1 is reduced to a different conjecture next, which we prove in one direction, and almost prove it in the other direction as well.

It is important for the rest of the paper to work locally at a point x in

$$X := \mathbb{C}^n.$$

In this case, we replace in the above B_F by the local Bernstein-Sato ideal $B_{F,x}$ of the germ of F at x, and we also propose the local version of Conjecture 1. It is known that

$$B_F = \bigcap_{x \in X} B_{F,x},$$

see [4, Corollary 6]. Thus, if we let $V(I)$ denote the zero locus of an ideal I,

$$V(B_F) = \bigcup_{x \in X} V(B_{F,x}),$$

and the local version implies the global version of Conjecture 1.

The relation with local systems is in two steps. First, we propose a generalization of the well-known result of Kashiwara [19] and Malgrange [24] which states that the roots of the classical Bernstein-Sato polynomial of a polynomial germ f give the

monodromy eigenvalues on the Milnor fiber. In this case, the cohomology of the Milnor fiber is packaged into Deligne's nearby cycles complex $\psi_f\mathbb{C}_X$. When $r \geq 1$, a generalization of Deligne's nearby cycles functor is the *Sabbah specialization functor*

$$\psi_F : D^b_c(X,\mathbb{C}) \to D^b_c(D,A),$$

where

$$D := \bigcup_{j=1}^{r} V(f_j)$$

is the union of the zero loci of the f_j,

$$A := \mathbb{C}[t_1, t_1^{-1}, \dots, t_r, t_r^{-1}],$$

and $D^b_c(.,R)$ is the bounded derived category of constructible sheaves in the analytic topology over a ring R. This functor has been introduced in [32]. The action of A on $\psi_F\mathbb{C}_X$ generalizes the monodromy of the Milnor fiber from the case $r = 1$.

For a point x in D, denote by

$$\mathrm{Supp}_x(\psi_F\mathbb{C}_X)$$

the support of $\psi_F\mathbb{C}_X$ at x as an A-module, see Definition 3.6. The ambient space of the support is the torus $(\mathbb{C}^*)^r$, with affine coordinate ring A. For our purposes, we have to take into account the possibility that some f_j do not vanish at x, and thus we are lead to define the *uniform support*

$$\mathrm{Supp}_x^{unif}(\psi_F\mathbb{C}_X) \quad \subset (\mathbb{C}^*)^r,$$

see Definition 3.20. Let

$$Exp : \mathbb{C}^r \longrightarrow (\mathbb{C}^*)^r$$

be the map $x \mapsto \exp(2\pi i x)$. The following generalizes the theorem of Kashiwara and Malgrange.

Conjecture 2.

$$Exp(V(B_{F,x})) = \bigcup_{y\in D \ near\ x} Supp_y^{unif}(\psi_F\mathbb{C}_X).$$

The union is taken over points $y \in D$ in a small open ball around x. However, one can take only the general points y of a fine enough stratification of the singular locus of D.

We show the following partial confirmation of Conjecture 2:

Theorem 1. *Assume all $f_j \in \mathbb{Q}[x_1, \dots, x_n]$ or, more generally, that $B_{F,x}$ is generated by polynomials with rational coefficients. Then*

$$Exp(V(B_{F,x})) \supset \bigcup_{y\in D \ near\ x} Supp_y^{unif}(\psi_F\mathbb{C}_X).$$

It is known that $B_{F,x}$ is generated by polynomials with rational coefficients if all f_j have rational coefficients, see [4, §4].

The method of proof of Theorem 1 is geometrical. However, it seems unlikely that the converse of Theorem 1 can be done without the use of $\mathcal{D}$-module theory.

We also make a significant step toward proving the converse of Theorem 1, see Proposition 6.9, where we show that it reduces to the following simple-to-state $\mathcal{D}$-module theoretic condition. Let $\mathcal{D}_X$ be the sheaf of holomorphic differential operators on X.

Conjecture 3. *If $\alpha \in V(B_{F,x})$ then, locally at x,*

$$\sum_{j=1}^{r}(s_j-\alpha_j)\mathcal{D}_X[s_1,\ldots,s_r]f_1^{s_1}\ldots f_r^{s_r} \not\equiv \mathcal{D}_X[s_1,\ldots,s_r]f_1^{s_1}\ldots f_r^{s_r}$$

modulo $\mathcal{D}_X[s_1,\ldots,s_r]f_1^{s_1+1}\ldots f_r^{s_r+1}$.

The following is a weaker version of Conjecture 1.

Theorem 2. *Assume Conjecture 2. If the polynomials f_j with $f_j(x)=0$ define mutually distinct reduced and irreducible hypersurface germs at x, then the set $Exp(V(B_{F,x}))$ is a finite union of torsion-translated subtori of $(\mathbb{C}^*)^r$.*

The relation of Bernstein-Sato ideals with local systems is completed by relating the latter with the Sabbah specialization. For a connected finite CW-complex M, let $L(M)$ denote the space of complex local systems of rank one on M. Then

$$L(M) = \mathrm{Hom}(H_1(M,\mathbb{Z}),\mathbb{C}^*).$$

Define the *characteristic variety of M* to be the subset $\mathcal{V}(M)$ of $L(M)$ consisting of local systems with non-trivial homology,

$$\mathcal{V}(M) := \{\mathcal{L}\in L(M) \mid H_k(M,\mathcal{L})\neq 0 \text{ for some } k\}. \tag{1}$$

There are other, more refined, (co)homology jump loci of M which can be defined, but we will not be concerned with them in this article.

For a point x in X, let $U_{F,x}$ be the complement of D in a small open ball centered at x,

$$U_{F,x} := \mathrm{Ball}_x - (\mathrm{Ball}_x \cap D).$$

There is a natural embedding of $L(U_{F,x})$ into the torus $(\mathbb{C}^*)^r$ induced by F.

Theorem 3. *If the polynomials f_j with $f_j(x)=0$ define mutually distinct reduced and irreducible hypersurface germs at x, then*

$$Supp_x(\psi_F\mathbb{C}_X) = \mathcal{V}(U_{F,x}).$$

We can define the *uniform characteristic variety* with respect to F, which we denote by $\mathcal{V}^{unif}(U_{F,x})$, such that it agrees with $\mathrm{Supp}_x^{unif}\psi_F\mathbb{C}_X$ via Theorem 3, see Definition 3.21.

Theorem 4. *Assume Conjecture 2. If the polynomials f_j with $f_j(x)=0$ define mutually distinct reduced and irreducible hypersurface germs at x, then*

$$Exp(V(B_{F,x})) = \bigcup_{y\in D \text{ near } x} \mathcal{V}^{unif}(U_{F,y}).$$

Theorem 4 implies immediately Theorem 2: $\mathcal{V}(U_{F,y})$ is a finite union of torsion-translated subtori by Libgober [22, Theorem 3.1], and hence the same is true for $\mathcal{V}^{unif}(U_{F,y})$. Thus, it is plausible that Conjecture 1 can be proved using Theorem 4.

Let us mention the connection with local Alexander modules. The cohomologies of the stalks of $\psi_F\mathbb{C}_X$ are the multi-variable local homology Alexander modules, as shown by Sabbah [32], see Proposition 3.12. In the special case when all the polynomials f_j are homogeneous, the cohomologies of the stalk at the origin of $\psi_F\mathbb{C}_X$ are the multi-variable universal homology Alexander modules introduced by Dimca-Maxim [12], see Proposition 3.24.

1.2. **The geometry of Bernstein-Sato ideals.** Next, information about uniform supports and characteristic varieties leads to better understanding of the question of what do zero loci of Bernstein-Sato ideals look like. In the case when all f_j are homogeneous polynomials, we give a formula which, assuming Conjecture 2, reduces the computation of $Exp(V(B_F))$ to a lower-dimensional, but possibly non-homogeneous, case, see Proposition 3.27. We also obtain:

Corollary 1. *Let $F = (f_1, \ldots, f_r)$ with $f_j \in \mathbb{C}[x_1, \ldots, x_n]$ irreducible and homogeneous of degree d_j defining mutually distinct hypersurfaces with $gcd(d_1, \ldots, d_r) = 1$. Let V be the complement in $\mathbb{P}^{n-1}$ of the union of the zero loci of f_j. If $\chi(V) \neq 0$, then*

$$\{d_1s_1 + \ldots + d_rs_r + k = 0\} \subset V(B_F)$$

for some $k \in \mathbb{Z}$.

It is tempting to conjecture that $k = n$ in Corollary 1. We do so below for hyperplane arrangments.

In the case of hyperplane arrangements, the homogenous reduction formula can be applied repeatedly to obtain precise formulas. Let $F = (f_1, \ldots, f_r)$ be such that f_j are linear forms defining mutually distinct hyperplanes. The following terminology is defined in section 5. For an edge W of the hyperplane arrangement $\prod_{j=1}^r f_j$, let F_W be the restriction in the sense of hyperplane arrangements of F to W. Let

$$F_W = \prod_{i=1}^{l_W} F_W^{(i)}$$

be a total splitting of F_W. If we set $F_W^{(i)} = (f_{1,W}^{(i)}, \ldots, f_{r,W}^{(i)})$, let

$$d_{j,W}^{(i)} := \deg f_{j,W}^{(i)}.$$

The following is an immediate consequence of Proposition 5.9 and Theorem 1:

Corollary 2. *Let $F = (f_1, \ldots, f_r)$ with $f_j \in \mathbb{C}[x_1, \ldots, x_n]$ linear forms defining mutually distinct hyperplanes. Then*

$$\bigcup_W V((t_1^{d_{1,W}^{(i)}} \ldots t_r^{d_{r,W}^{(i)}} - 1 \mid i = 1, \ldots, l_W)) \quad \subset \quad Exp(V(B_F)),$$

where the union is over the edges W of the hyperplane arrangement $\prod_{j=1}^{r} f_j$. Assuming Conjecture 2, equality holds.

By specializing $F = (f_1, \dots, f_r)$ to $\prod_{j=1}^{r} f_r$ in the above Corollary, see also Theorem 8 below, we obtain the following. Let f be a hyperplane arrangement, f_W the restriction to the edge W, and $f_W = \prod_{i=1}^{l_W} f_W^{(i)}$ a total splitting of f_W. Let $d_W^{(i)} = \deg f_W^{(i)}$. Denote by b_f the classical one-variable Bernstein-Sato polynomial of f, and by $M_{f,x}$ the Milnor fiber of f at x. With this notation:

Corollary 3. *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a reduced hyperplane arrangement. Then $Exp(V(b_f))$, which equals the set of all eigenvalues of the monodromy on $H^\bullet(M_{f,x}, \mathbb{C})$ for x ranging over $f^{-1}(0)$, is a combinatorial invariant equal to*

$$\bigcup_W V(t^{d_W^{(i)}} - 1 \mid i = 1, \dots, l_W),$$

where the union is over the edges W of f.

In contrast, U. Walther has announced that the Bernstein-Sato polynomial b_f of a hyperplane arrangement is not a combinatorial invariant. A different proof of Corollary 3 involving [21, Theorem 3.1] was noticed and communicated to us by A. Libgober.

The following is a multi-variable generalization of [8, Conjecture 1.2]. This simple-looking, yet difficult-to-prove, statement has implications for the Multi-Variable Strong Monodromy Conjecture, see Theorem 6 below.

Conjecture 4. *If $f_1, \dots, f_r$ are central hyperplane arrangements in $\mathbb{C}^n$, not necessarily reduced, of degree d_j, and $\prod_{j=1}^{r} f_j$ is a central essential indecomposable hyperplane arrangement, then*

$$\{d_1 s_1 + \ldots + d_r s_r + n = 0\} \subset V(B_F).$$

1.3. **Multi-Variable Monodromy Conjecture.** We discuss the relation between multi-variable topological zeta functions on one hand, and Sabbah specialization complexes and Bernstein-Sato ideals, on other hand.

Let $F = (f_1, \dots, f_r)$ with $f_j \in \mathbb{C}[x_1, \dots, x_n]$. We keep the notation from 1.1. Let $\mu : Y \to X$ be a log resolution of $\prod_j f_j$. Let E_i for $i \in S$ be the collection of irreducible components of the zeros locus of $(\prod_j f_j) \circ \mu$. Let $a_{i,j}$ be the order of vanishing of f_j along E_i, and let k_i be the order of vanishing of the determinant of the Jacobian of μ along E_i. For $I \subset S$, let $E_I^\circ = \cap_{i \in I} E_i - \cup_{i \in S-I} E_i$. With this notation, the *topological zeta function* of $F = (f_1, \dots, f_r)$ is

$$Z_F^{top}(s_1, \dots, s_r) := \sum_{I \subseteq S} \chi(E_I^\circ) \cdot \prod_{i \in I} \frac{1}{a_{i,1}s_1 + \ldots a_{i,r}s_r + k_i + 1}.$$

This expression is independent of the choice of log resolution. Define

$$PL(Z_F^{top}(s_1, \dots, s_r))$$

to be the polar locus in $\mathbb{C}^r$.

The following is the *Topological Multi-Variable Monodromy Conjecture*, slightly different than phrased by Loeser, see [26, 23]:

Conjecture 5.

$$Exp(PL(Z_F^{top})) \subset \bigcup_{x \in D} Supp_x^{unif}(\psi_F \mathbb{C}_X).$$

When $r = 1$, this is the Monodromy Conjecture of Igusa-Denef-Loser saying that poles of the topological zeta function give eigenvalues of the Milnor monodromy.

We call the following the *Topological Multi-Variable Strong Monodromy Conjecture*:

Conjecture 6.

$$PL(Z_F^{top}) \subset V(B_F).$$

Conjecture 6 implies Conjecture 5 if we believe Conjecture 2, hence the adjective "strong".

We prove Conjecture 5 in the following case:

Theorem 5. *If each f_j define a (possibly nonreduced) hyperplane arrangement in $\mathbb{C}^n$, then Conjecture 5 holds.*

Besides the hyperplane arrangements case, Conjecture 5 is also known for plane curves, see Nicaise [26].

We cannot prove Conjecture 6 for hyperplane arrangements, but we reduce it to Conjecture 4:

Theorem 6. *If each f_j define a (possibly-nonreduced) hyperplane arrangement in $\mathbb{C}^n$ and if Conjecture 4 holds for the restriction*

$$F_W = (f_{j,W} \mid f_j(W) = 0, j \in \{1, \dots, r\})$$

of the hyperplane arrangements to any dense edge W of $\prod_{j=1}^r f_j$, then Conjecture 6 holds for F.

Both Theorems 5 and 6 were proved for $r = 1$ in [8], and here we follow a similar strategy.

On a different note, there has been recent interest in zeta functions attached to differential forms and possible connections with monodromy-type invariants, see Némethi-Veys [25]. Let $dx = dx_1 \wedge \ldots \wedge dx_n$ and let $\omega = f_{r+1} dx$ be an n-form on X. Define

$$Z_F^{top,\, \omega}(s_1, \dots, s_r)$$

in a similiar fashion as $Z_F^{top}(s_1, \dots, s_r)$, but with k_i replaced by $ord_{E_i}\omega$. Note that

$$Z_F^{top,\, dx}(s_1, \dots, s_r) = Z_F^{top}(s_1, \dots, s_r).$$

One can ask *what would a Monodromy Conjecture predict for $Z_F^{top,\, \omega}$?* See [25, 1.2] for a discussion. We propose an answer which is natural from the point of view of the Sabbah specialization complex. Clearly

$$Z_{(f_1,\dots,f_r)}^{top,\, f_{r+1}dx}(s_1, \dots, s_r) = Z_{(f_1,\dots,f_{r+1})}^{top}(s_1, \dots, s_r, 1).$$

Since the polar locus of a specialization of a rational function is included in the specialization of the polar locus of the same rational function, we see that the right (Strong) Monodromy Conjecture for forms is a specialization of Conjectures 5 and 6:

Theorem 7. *(a) (Topological Multi-Variable Monodromy Conjecture for Forms.) Assume Conjecture 5 holds for F. Then*

$$Exp(PL(Z^{top,\, f_r dx}_{(f_1,\dots,f_{r-1})})) \subset \bigcup_{x\in D} Supp^{unif}_x(\psi_F\mathbb{C}_X)\cap V(t_r-1).$$

(b) (Topological Multi-Variable Strong Monodromy Conjecture for Forms.) Assume Conjecture 6 holds for F. Then

$$PL(Z^{top,\, f_r dx}_{(f_1,\dots,f_{r-1})}) \subset V(B_F)\cap V(s_r-1).$$

Again, part (b) implies part (a) if we believe Conjecture 2.

One can ask how natural is to specialize the Monodromy Conjecture. We define later what it means to specialize F to another collection G of possibly fewer polynomials, see Definition 3.29. For example, $F=(f_1,\dots,f_r)$ specializes to $G=(f_1,\dots,f_{r-1})$, and it also specializes to $\prod_j f_j$. We show the following naturality with respect to specializations of the Monodromy Conjecture:

Theorem 8. *Assume that Conjecture 5 holds for a given F. If F specializes to G, then Conjecture 5 also holds for G.*

At the moment we cannot conclude the same thing for the Strong Monodromy Conjecture.

It is a standard procedure to adjust statements involving topological zeta functions to obtain statements involving: local topological zeta functions, (local) p-adic zeta functions, and, more generally, (local) motivic zeta functions. For brevity, we shall skip this discussion.

1.4. **Computational aspects.** Bernstein-Sato ideals are amenable to computer calculations, albeit expensive ones. Thus, by transfer, Conjecture 2 would provide algorithms to compute the characteristic varieties appearing on the right-hand side of the equality in Theorem 4. There are no other known algorithms for characteristic varieties applicable in general. Note that Berstein-Sato ideals are essential for the current algorithms computing cohomology of local systems on complements of projective hypersurfaces, see Oaku-Takayama [27].

We also push a bit the computability boundary. We provide a general structural result on $Exp(V(B_F))$ in Proposition 6.6. This result, combined with current algorithms for other ideals of Bernstein-Sato type related to B_F, can be used to compute $Exp(V(B_F))$ even when B_F is intractable, see Example 7.1.

1.5. **Acknowledgement.** We would like to thank V. Levandovskyy and U. Walther for their help with computing examples. For computations of the Bernstein-Sato ideals in this paper we used the library `dmod.lib` in SINGULAR [10, 20]. We would also like to thank A. Dimca, A. Libgober, L. Maxim, M. Schulze, and W. Veys for

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1.6. **Notation.** All algebraic varieties are assumed to be over the complex number field. A variety is not assumed to be irreducible.

2. Characteristic varieties

2.1. **Local systems and characteristic varieties.** Let M be a connected finite CW-complex of dimension n. Let $L(M)$ be the group of rank one complex local systems on M. We can identify

$$L(M) = \mathrm{Hom}(\pi_1(M), \mathbb{C}^*) = \mathrm{Hom}(H_1(M, \mathbb{Z}), \mathbb{C}^*) = H^1(M, \mathbb{C}^*).$$

Consider the ring

$$B := \mathbb{C}[H_1(M, \mathbb{Z})].$$

Then $L(M)$ is an affine variety with affine coordinate ring equal to B.

Example 2.2. If U denotes the complement in a small open ball centered at a point x in $\mathbb{C}^n$ of r mutually distinct analytically irreducible hypersurface germs, then $H_1(U, \mathbb{Z}) = \mathbb{Z}^r$ is generated by the classes of small loops around the branches, and $L(U) = (\mathbb{C}^*)^r$, see [11, (4.1.5)].

Example 2.3. If V denotes the complement in $\mathbb{P}^{n-1}$ of r mutually distinct reduced and irreducible hypersurfaces of degrees $d_1, \dots, d_r$, then

$$H_1(V, \mathbb{Z}) = \left[\bigoplus_{j=1}^{r} \mathbb{Z} \cdot \gamma_j \right] / (d_1\gamma_1 + \ldots + d_r\gamma_r),$$

where γ_j is the class of a small loop centered at a general point on the j-th hypersurface, see [11, (4.1.3)].

Let M^{ab} be the universal abelian cover of M. In other words, M^{ab} is the cover of M given by the kernel of the natural abelianization map

$$ab : \pi_1(M) \to H_1(M, \mathbb{Z}).$$

Definition 2.4. *The characteristic variety of M is the subset*

$$\mathcal{V}(M) := \bigcup_k \mathit{Supp}(H_k(M^{ab}, \mathbb{C}))$$

of $L(M)$, where $\mathit{Supp}(H_k(M^{ab}, \mathbb{C}))$ is the support of the B-module $H_k(M^{ab}, \mathbb{C})$.

This definition agrees with the definition (1) from the Introduction, see [29, Theorem 3.6]:

Theorem 2.5. *$\mathcal{V}(M)$ is the set of local systems of rank one on M with non-trivial homology.*

3. Sabbah specialization and local systems

In this section we write down some properties of the Sabbah specialization complexes. We also prove Theorems 3, 4, as well as the homogeneous reduction formula mentioned in 1.2.

For a ring R and a variety X, let $D^b_c(X, R)$ denote the bounded derived category of R-constructible sheaves on the underlying analytic variety of X.

3.1. **Sabbah specialization.** Let

$$F = (f_1, \ldots, f_r)$$

with $f_j \in \mathbb{C}[x_1, \ldots, x_n]$. Let

$$X = \mathbb{C}^n, \qquad D_j = V(f_j) \subset X, \qquad D = \cup_{j=1}^r D_j, \qquad U = X - D.$$

Let $S = \mathbb{C}^r$, $S^* = (\mathbb{C}^*)^r$, and denote by $\widetilde{S^*}$ the universal cover of S^*. We denote the affine coordinate ring of S^* by

$$A = \mathbb{C}[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}].$$

Consider the following diagram of fibered squares of natural maps:

$$\begin{tikzcd} D \arrow[r, hook, "i_D"'] & X \arrow[d, "F"'] & U \arrow[l, hook', "j"'] \arrow[d, "F_U"'] & \widetilde{U} \arrow[l, "p"'] \arrow[d, "F_{\widetilde{U}}"'] \\ & S & S^* \arrow[l, hook', "j_S"] & \widetilde{S^*} \arrow[l, "p_S"] \end{tikzcd}$$

Definition 3.2. *The Sabbah specialization functor of F is*

$$\psi_F = i_D^* Rj_* Rp_! (j \circ p)^* : D^b_c(X, \mathbb{C}) \to D^b_c(D, A).$$

We call $\psi_{\mathcal{F}} \mathbb{C}_X$ the Sabbah specialization complex.

Remark 3.3. (1) This definition is slightly different than [32, 2.2.7] where it is called *the nearby Alexander complex*. To obtain the definition in *loc. cit.*, one has to restrict further to $\cap_j D_j$. The constructibility ψ_F over A follows as in *loc. cit.*

(2) This definition is also slightly different than the one in [26], where $Rp_!$ is replaced by Rp_*.

(3) When $r = 1$, $\psi_f \mathcal{F}$ as defined here equals $\psi_f \mathcal{F}[-1]$ as defined by Deligne, see [6, p.13].

A different expression for Sabbah specialization is as follows. Let

$$\mathcal{L} = R(p_S)_! \mathbb{C}_{\widetilde{S^*}} = (p_S)_! \mathbb{C}_{\widetilde{S^*}}.$$

This is the rank-one local system of free A-modules on S^* corresponding to the isomorphism

$$\pi_1(S^*) \to \mathbb{Z}^r = \pi_1(S^*).$$

Define

$$\mathcal{L}^F := F_U^* \mathcal{L}.$$

This is a local system of A-modules on U. The following is essentially a particular case of [32, 2.2.8] in light of Remark 3.3 (1):

Lemma 3.4. *(a)* $\psi_F \mathcal{F} = i_D^* Rj_*(j^*\mathcal{F} \otimes_{\mathbb{C}_U} \mathcal{L}^F)$.

(b) In particular, $\psi_F \mathbb{C}_X = i_D^* Rj_* \mathcal{L}^F$.

Proof. By the projection formula [32, 2.1.3], $Rj_* Rp_! p^* j^* \mathcal{F} = Rj_*(j^*\mathcal{F} \otimes_{\mathbb{C}_U} Rp_! \mathbb{C}_{\widetilde{U}})$. On the other hand, we have $Rp_! \mathbb{C}_{\widetilde{U}} = Rp_! F_{\widetilde{U}}^* \mathbb{C}_{\widetilde{S^*}} = F_U^* Rp_! \mathbb{C}_{\widetilde{S^*}} = \mathcal{L}^F$. □

3.5. **Multi-variable monodromy zeta function.** We recall Sabbah's multi-variable generalization of A'Campo's formula for the monodromy zeta function.

Definition 3.6. *For an A-module G, we denote by $Supp(G)$ the support of G in $S^* = Spec\, A$. For an A-constructible sheaf $\mathcal{G}$ on X and a point x in X, the support of $\mathcal{G}$ at x is the support of the stalk:*

$$Supp_x(\mathcal{G}) := Supp(\mathcal{G}_x) \subset S^*$$

The support at the point x of a complex $\mathcal{G} \in D_c^b(D, A)$ is

$$Supp_x(\mathcal{G}) := \bigcup_k Supp_x(H^k(\mathcal{G})) \subset S^*.$$

Definition 3.7. *The multi-variable monodromy zeta function of $\mathcal{G} \in D_c^b(D, A)$ at the point x is defined to be the cycle*

$$\zeta_x(\mathcal{G}) := \sum_P \chi_x(\mathcal{G}_P) \cdot V(P),$$

where the sum is over prime ideals P of A of height one among those such that their zero locus $V(P) \subset Supp_x(\mathcal{G})$, $\mathcal{G}_P$ is the localization of $\mathcal{G}$ at the prime ideal P, and χ_x is the stalk Euler characteristic.

Codimension-one cycles on S^* can be viewed as rational functions in $t_1, \ldots, t_r$ up to multiplication by a monomial. We will use the rational function notation for the multi-variable monodromy zeta function of Sabbah specialization complex:

$$\zeta_x(\psi_F \mathbb{C}_X)(t_1, \ldots, t_r) := \zeta_x(\psi_F \mathbb{C}_X).$$

Let $\mu : Y \to X$ be a log resolution of $\prod_j f_j$. Let E_i for $i \in S$ be the collection of irreducible components of the zeros locus of $(\prod_j f_j) \circ \mu$. Let $a_{i,j}$ be the order of vanishing of f_j along E_i, and let k_i be the order of vanishing of the determinant of the Jacobian of μ along E_i. For $I \subset S$, let $E_I^\circ = \cap_{i \in I} E_i - \cup_{i \in S-I} E_i$. The following is Sabbah's generalization of A'Campo's formula, see [32, 2.6.2]:

Theorem 3.8. *If $f_j(x) = 0$ for all j, then*

$$\zeta_x(\psi_F \mathbb{C}_X)(t_1, \ldots, t_r) = \prod_{i \in I \text{ with } \mu(E_i) = x} (t_1^{a_{i,1}} \ldots t_r^{a_{i,r}} - 1)^{-\chi(E_i^\circ)}.$$

Corollary 3.9. *If f_j are homogeneous polynomials of degree d_j for all j, and $\chi(V) \neq 0$, where $V = \mathbb{P}^{n-1} - \mathbb{P}(D)$, then*

$$V(t_1^{d_1} \ldots t_r^{d_r} - 1) \subset Supp_0(\psi_F \mathbb{C}_X).$$

Proof. In this case, one can take a log resolution $\mu : Y \to X$ which factors through the blow-up at 0, such that the strict transform E of the blow-up exceptional divisor E' is the only exceptional divisor of Y mapping to 0, and such that μ is an isomorphism outside D. In this case, $E' = \mathbb{P}^{n-1}$, $E^\circ = V$. Then, by Theorem 3.8,

$$\zeta_0(\psi_F\mathbb{C}_X)(t_1,\ldots,t_r) = (t_1^{d_1}\ldots t_r^{d_r}-1)^{-\chi(V)},$$

and the conclusion follows. □

For $r = 1$ one recovers a well-known formula for the monodromy zeta function of a homogeneous polynomial, see [11, p.108].

3.10. **Local Alexander modules.** Let x be a point in D and $i_x : \{x\} \to D$ the natural inclusion. Let Ball_x be a small open ball centered at x in X, and let

$$U_{F,x} = \mathrm{Ball}_x - D.$$

Let $U_{F,x}^{ab}$ be the universal abelian cover of $U_{F,x}$, and let

$$B = \mathbb{C}[H_1(U_{F,x},\mathbb{Z})].$$

Let

$$\widetilde{U_{F,x}} := U_{F,x} \times_{S^*} \widetilde{S^*}.$$

Consider the commutative diagram of fibered squares

$$\begin{tikzcd} \mathrm{Ball}_x \arrow[d, "F"'] & U_{F,x} \arrow[l, hook'] \arrow[d] & \widetilde{U_{F,x}} \arrow[l, "p"'] \arrow[d] \\ S & S^* \arrow[l, hook'] & \widetilde{S^*}. \arrow[l] \end{tikzcd}$$

Definition 3.11. *The k-th local homology Alexander module of F at x is the A-module*

$$H_k(\widetilde{U_{F,x}},\mathbb{C}).$$

The k-th local homology Alexander module of $U_{F,x}$ is the B-module

$$H_k(U_{F,x}^{ab},\mathbb{C}).$$

The following is essentially a particular case of [32, 2.2.5]:

Proposition 3.12. *The k-th cohomology of the stalk of the Sabbah specialization complex is isomorphic as an A-module, up to the switch between of the action of t_j with that of t_j^{-1}, to the k-th local homology Alexander module of F at x:*

$$H^k(i_x^*\psi_F\mathbb{C}_X) \cong_A H_k(\widetilde{U_{F,x}},\mathbb{C}).$$

Proof. By Lemma 3.4, $H^k(i_x^*\psi_F\mathbb{C}_X) = (R^kj_*\mathcal{L}^F)_x$. The stalk of the sheaf $R^kj_*\mathcal{L}^F$ at x equals the stalk of the presheaf $V \mapsto H^k(U\cap V,\mathcal{V})$. Hence

$$H^k(i_x^*\psi_F\mathbb{C}_X) = H^k(U_{F,x},\mathcal{L}^F). \tag{2}$$

We have

$$\begin{aligned} H_k(\widetilde{U_{F,x}},\mathbb{C}) &= H_c^{2n-k}(\widetilde{U_{F,x}},\mathbb{C}) = H^{2n-k}(Ra_!p_!\mathbb{C}_{\widetilde{U_{F,x}}}) \\ &= H_c^{2n-k}(U_{F,x},\mathcal{L}^F) = H^{2n-k}(Ra_!\mathcal{L}^F), \end{aligned}$$

where a is the map to a point. On the other hand, letting $(\mathcal{L}^F)^\vee$ be the A-dual local system of $\mathcal{L}^F$, and D_A be the Verdier duality functor, we have

$$\begin{aligned} H^{2n-k}(Ra_!\mathcal{L}^F) &= H^{n-k}(Ra_! D_A((\mathcal{L}^F)^\vee[n])) = H^{n-k}(D_A Ra_*(\mathcal{L}^F)[n]) \\ &= D_A H^k(Ra_*(\mathcal{L}^F)^\vee) = D_A H^k(U_{F,x},(\mathcal{L}^F)^\vee). \end{aligned}$$

The last A-module is isomorphic, after the change of t_j with t_j^{-1}, with $H^k(U_{F,x},\mathcal{L}^F)$. ☐

Lemma 3.13. *If the polynomials f_j with $f_j(x)=0$ define mutually distinct reduced and irreducible hypersurface germs at x, then*

$$\widetilde{U_{F,x}} = U^{ab}_{F,x} \quad \textit{and} \quad B = A/(t_j - 1 \mid f_j(x) \neq 0).$$

Proof. The second assertion follows from Example 2.2. For the first assertion, it is enough to show that $\widetilde{U_{F,x}}$ corresponds to the kernel of the abelianization map

$$\pi_1(U_{F,x}) \to H_1(U_{F,x},\mathbb{Z}).$$

By definition, $\widetilde{U_{F,x}}$ is given by the kernel of the composition

$$\pi_1(U_{F,x}) \xrightarrow{F_*} \pi_1(S^*) = \mathbb{Z}^r \xrightarrow{id} H_1(S^*,\mathbb{Z}) = \mathbb{Z}^r.$$

Since the codomain is abelian, it is enough to show that the natural direct image

$$F_* = ((f_1)_*,\dots,(f_r)_*) : H_1(U_{F,x},\mathbb{Z}) \longrightarrow H_1(S^*,\mathbb{Z})$$

is injective. By Example 2.2, $H_1(U_{F,x},\mathbb{Z})$ is free abelian generated by the classes of loops γ_j centered a general point of $\mathrm{Ball}_x \cap D_j$ for those j such that $f_j(x)=0$. Let δ_j be a generator for the first homology of the j-th copy of $\mathbb{C}^*$ in S^*. Both assertions of the Lemma follow then from the fact that

$$f_j(\gamma_{j'}) \sim \begin{cases} 0 & \text{if } f_j(x) \neq 0, \text{ or } j \neq j', \\ \delta_j & \text{if } f_j(x) = 0 \text{ and } j = j'. \end{cases}$$

Indeed, if $f_j(x)\neq 0$, then $\gamma_{j'}$ can be chosen such that $f_j(\gamma_{j'})$ is a loop homologically equivalent to 0. If $j \neq j'$, then $\gamma_{j'}$ can be chosen such that $f_j(\gamma_{j'})$ is point. If $f_j(x)=0$ and $j=j'$, then γ_j can be chosen to intersect at most once every fiber of f_j, hence $f_j(\gamma_j)$ is homologically equivalent to δ_j. ☐

From Proposition 3.12 and Lemma 3.13 we obtain:

Corollary 3.14. *If the polynomials f_j with $f_j(x)=0$ define mutually distinct reduced and irreducible hypersurface germs at x, then the k-th cohomology of the stalk of $\psi_F\mathbb{C}_X$ at x is the k-th local homology Alexander module of $U_{F,x}$. More precisely, the action of A on $H^k(i_x^*\psi_F\mathbb{C}_X)$ factors through the action of B, and*

$$H^k(i_x^*\psi_F\mathbb{C}_X) \cong H_k(U^{ab}_{F,x},\mathbb{C})$$

as B-modules after replacing on the right-hand side the t_j-action with the t_j^{-1}-action.

Remark 3.15. In the case when one the polynomials f_r is nonsingular outside $\cup_{j=1}^{r-1}D_j$, or, when f_r is a generic linear polynomial through x, more information is available about the local Alexander modules from [32, 2.6.3 and 2.6.4].

3.16. Proof of Theorem 3. In this case,

$$L(U_{F,x}) = \operatorname{Spec} B = \bigcap_{j : f_j(x) \neq 0} V(t_j - 1) \subset S^*$$

by Example 2.2. The claim follows from Corollary 3.14 and Theorem 2.5. $\square$

3.17. **Uniform support.** Even if the polynomials f_j vanishing at x do not define mutually distinct reduced and irreducible hypersurface germs at x, the injectivity part in the proof of Lemma 3.13, together with Proposition 3.12, prove the following:

Lemma 3.18. *The action of A on $H^k(i_x^* \psi_F \mathbb{C}_X)$ factors through the action of*

$$A/(t_j - 1 \mid f_j(x) \neq 0).$$

As a consequence, for a point $x \in D$, the support $\operatorname{Supp}_x \psi_F \mathbb{C}_X$ lies in a sub-torus of $S^* = (\mathbb{C}^*)^r$ of codimension exactly the number of polynomials f_j with $f_j(x) \neq 0$. More precisely, let

$$\mathbb{T}_x = \bigcap_{j : f_j(x) \neq 0} V(t_j - 1) \subset S^*.$$

Then

$$\operatorname{Supp}_x \psi_F \mathbb{C}_X \subset \mathbb{T}_x.$$

Let r_x be the codimension of $\mathbb{T}_x$ in S^*.

Definition 3.19. *The F-natural splitting of S^* is the splitting*

$$S^* = \mathbb{T}_x \times (\mathbb{C}^*)^{r_x}$$

compatible with the splitting

$$\{j \mid f_j(x) = 0\} \cup \{j \mid f_j(x) \neq 0\}.$$

Definition 3.20. *The uniform support at x of $\psi_F \mathbb{C}_X$ is*

$$Supp_x^{unif} \psi_F \mathbb{C}_X := (Supp_x \psi_F \mathbb{C}_X) \times (\mathbb{C}^*)^{r_x} \subset S^*,$$

the last inclusion being induced by the F-natural splitting of S^*. By definition, the uniform support coincides with the usual support when $\mathbb{T}_x$ is empty, or in other words, when all f_j vanish at x.

Similarly, consider the characteristic variety $\mathcal{V}(U_{F,x})$. This is a subvariety of the space of rank one local systems $L(U_{F,x})$, and

$$L(U_{F,x}) \subset \mathbb{T}_x,$$

with $\mathbb{T}_x$ as above. We have an equality $L(U_{F,x}) = \mathbb{T}_x$ if the germs f_j which vanish at x define mutually distinct reduced and irreducible hypersurface germs.

Definition 3.21. *The uniform characteristic variety of F at x is*

$$\mathcal{V}^{unif}(U_{F,x}) := \mathcal{V}(U_{F,x}) \times (\mathbb{C}^*)^{r_x} \subset S^*,$$

the last inclusion being induced by the F-natural splitting of S^*. By definition, the uniform characteristic variety coincides with the usual characteristic variety when $\mathbb{T}_x$ is empty, or in other words, when all f_j vanish at x.

3.22. Proof of Theorem 4. It follows from Theorem 3 and the definition of uniform support and uniform characteristic variety above, together with the assumption that Conjecture 2 holds. □

3.23. **Homogeneous polynomials.** Assume now that f_j are homogeneous polynomials for all j. We show first that $i_0^*\psi_F\mathbb{C}_X$ recovers the *universal Alexander modules* of Dimca-Maxim [12, §5]. Let

$$V = \mathbb{P}^{n-1} - \bigcup_{j=1}^{r} \mathbb{P}(D_j).$$

Let d_j be the degree of f_j. Let V^{ab} be the universal abelian cover of V. Then $H_k(V^{ab}, \mathbb{C})$ admits an action of

$$B := \mathbb{C}[H_1(V, \mathbb{Z})].$$

If f_j are mutually distinct irreducible homogeneous polynomials, which is the situation considered in [12],

$$B = A/(t_1^{d_1} \cdots t_r^{d_r} - 1),$$

and

$$L(V) = V(t_1^{d_1} \cdots t_r^{d_r} - 1) \subset S^*,$$

see Example 2.3.

Proposition 3.24. *If f_j are irreducible homogeneous polynomials of degree d_j defining mutually distinct hypersurfaces with $gcd(d_1, \dots, d_r) = 1$, then the action of A on $i_0^*\psi_F\mathbb{C}_X$ factorizes through B, and*

$$H^k(i_0^*\psi_F\mathbb{C}_X) \cong H_k(V^{ab}, \mathbb{C})$$

as B-modules after replacing on the right-hand side the t_j-action with the t_j^{-1}-action.

Proof. Consider $U_{F,0}$, the complement in a small open ball at the origin of D. The natural projectivization map $U_{F,0} \to V$ has fibers diffeomorphic with $\mathbb{C}^*$ and is a deformation retract of the restriction of the tautological line bundle of $\mathbb{P}^{n-1}$ to V. Since $gcd(d_1, \dots, d_r) = 1$, the Picard group of V is trivial. Hence, topologically,

$$U_{F,0} \approx \mathbb{C}^* \times V.$$

Fix a section $\sigma : V \to U_{F,0}$. First, we show that via this section

$$V^{ab} \approx V \times_{U_{F,0}} U_{F,0}^{ab}.$$

The cover on the right-hand side is given by the kernel of the composition

$$\pi_1(V) \xrightarrow{\sigma_*} \pi_1(U_{F,0}) \xrightarrow{ab} H_1(U_{F,0}, \mathbb{Z}).$$

The cover on the left-hand side is given by the kernel of

$$\pi_1(V) \xrightarrow{ab} H_1(V, \mathbb{Z}).$$

Hence, it is enough to show that the map

$$\sigma_* : H_1(V, \mathbb{Z}) \longrightarrow H_1(U_{F,0}, \mathbb{Z})$$

is injective. By assumption, both groups are free abelian of rank $r-1$ and, respectively, r. Hence σ_* is compatible with the Künneth decomposition

$$H_1(U_{F,0},\mathbb{Z}) = H_1(V,\mathbb{Z}) \oplus \mathbb{Z},$$

by the naturality of the Künneth decomposition via cross products, and the injectivity follows. This also shows that A acts on $H_k(V^{ab},\mathbb{C})$ via the surjection

$$A = \mathbb{C}[H_1(U_{F,0},\mathbb{Z})] \longrightarrow B = \mathbb{C}[H_1(V,\mathbb{Z})] = A/(t_1^{d_1}\dots t_r^{d_r}-1)$$

induced by σ_*.

Now the Proposition follows from Corollary 3.14 and the fact that V^{ab} is a deformation retract of

$$U_{F,0}^{ab} \approx (\mathbb{C}^*)^{ab} \times V^{ab} \approx \mathbb{C}\times V^{ab}.$$

□

Proposition 3.25. *If f_j are irreducible homogeneous polynomials of degree d_j defining mutually distinct hypersurfaces with $gcd(d_1,\dots,d_r)=1$, then, in S^*:*

$$Supp_0(\psi_F\mathbb{C}_X) = \mathcal{V}(V) \subset L(V) = V(t_1^{d_1}\cdots t_r^{d_r}-1).$$

The four sets are equal if, in addition, $\chi(V)\neq 0$.

Proof. The first equality is new. It follows from Proposition 3.24 and Theorem 2.5. If $\chi(V)\neq 0$, the equality follows from Corollary 3.9. □

Remark 3.26. In the case when one of the homogeneous polynomials is a generic linear form, more information is available about $\mathcal{V}(V)$, see [12, Theorem 3.6] and see also Remark 3.15.

For a point $y\in X=\mathbb{C}^n$ different than the origin, let $[y]$ denote the point in $\mathbb{P}^{n-1}$ with homogeneous coordinates given by y. We denote by

$$U_{F,[y]}\subset \mathbb{P}^{n-1}$$

the complement in a small ball around $[y]$ of $\mathbb{P}(D)$. Note that if we consider an affine space neighborhood of $[y]$,

$$[y]\in\mathbb{A}^{n-1}\subset\mathbb{P}^{n-1},$$

then

$$U_{F,[y]} = U_{F_{|\mathbb{A}^{n-1}},[y]}.$$

There is a homotopy equivalence

$$U_{F,y}\approx_{ht} U_{F,[y]}.$$

Hence

$$\mathcal{V}(U_{F,y}) = \mathcal{V}(U_{F,[y]}) = \mathcal{V}(U_{F_{|\mathbb{A}^{n-1}},[y]})\subset\mathbb{T}_y,$$

with $\mathbb{T}_y$ as in 3.17. Moreover,

$$\mathcal{V}^{unif}(U_{F,y}) = \mathcal{V}^{unif}(U_{F_{|\mathbb{A}^{n-1}},[y]})\subset S^*$$

because the F-natural splitting $S^* = T_y \times (\mathbb{C}^*)^{r_y}$ is the same as the $F_{|\mathbb{A}^{n-1}}$-natural splitting. We define:

$$\mathcal{V}^{unif}(U_{F,[y]}) := \mathcal{V}^{unif}(U_{F,y}) = \mathcal{V}^{unif}(U_{F_{|\mathbb{A}^{n-1}},[y]}).$$

The following would reduce the computation of $Exp(V(B_F))$ to a lower-dimensional, possibly non-homogeneous, case:

Proposition 3.27. *Assume Conjecture 2. If the polynomials f_j are homogeneous of degree d_j and define mutually distinct reduced and irreducible hypersurfaces, and if $gcd(d_1, \dots, d_j) = 1$, then*

$$Exp(V(B_F)) = \mathcal{V}(V) \cup \bigcup_{[y]\in\mathbb{P}^{n-1}} \mathcal{V}^{unif}(U_{F,[y]}).$$

If, in addition, $\chi(V) \neq 0$, then $\mathcal{V}(V) = L(V) = V(t_1^{d_1} \cdots t_r^{d_r} - 1)$.

Proof. Since F is a collection of homogeneous polynomials and

$$V(B_F) = \bigcup_{x\in X} V(B_{F,x}),$$

we have by Conjecture 2 that

$$Exp(V(B_F)) = Exp(V(B_{F,0})).$$

The conclusion holds for $Exp(V(B_{F,0}))$ because of Theorem 4, Proposition 3.25, and the above discussion for $y \neq 0$. ☐

3.28. **Specialization of polynomial maps.** Next, we address the question of what happens with the support of $\psi_F \mathbb{C}_X$ under specialization of the map F in the following sense:

Definition 3.29. *We say that $F = (f_1, \dots, f_r)$ specializes to $G = (g_1, \dots, g_{r'})$, if f_j and $g_{j'}$ are in $\mathbb{C}[x_1, \dots, x_n]$, $r \geq r'$, and G is the composition $F' \circ F$ where*

$$F' = (f'_1, \dots, f'_{r'}) : \mathbb{C}^r \longrightarrow \mathbb{C}^{r'}$$

is such that $f'_{j'}$ are monomials and the induced map on tori $(\mathbb{C}^*)^r \to (\mathbb{C}^*)^{r'}$ is surjective.

We will use the notation:

$$S = \mathbb{C}^r, \quad S' = \mathbb{C}^{r'}, \quad S^* = (\mathbb{C}^*)^r, \quad (S')^* = (\mathbb{C}^*)^{r'}.$$

In the situation of Definition 3.29, the pull-back of local systems defines an inclusion of spaces of rank one local systems

$$L((S')^*) = (S')^* \hookrightarrow L(S^*) = S^*.$$

There is a corresponding inclusion of tangent spaces at identity

$$S' \hookrightarrow S$$

recovering the previous inclusion via $Exp(.)$. Let

$$\phi : A = \mathbb{C}[t_1, t_1^{-1}, \dots, t_r, t_r^{-1}] \twoheadrightarrow A' = \mathbb{C}[u_1, u_1^{-1}, \dots, u_{r'}, u_{r'}^{-1}]$$

be the corresponding surjection of the coordinate rings of S^* and $(S')^*$. Let D' be the union of the zero loci of $g_{j'}$.

The following is a consequence of [32, 2.3.8]:

Proposition 3.30. *If the polynomial map F specializes to G, then $Supp_x^{unif}(\psi_F\mathbb{C}_X)$ specializes to $Supp_x^{unif}(\psi_G\mathbb{C}_X)$ for all $x \in D'$, that is*

$$Supp_x^{unif}(\psi_F\mathbb{C}_X) \cap (S')^* = Supp_x^{unif}(\psi_G\mathbb{C}_X).$$

Proof. Recall that

$$\mathrm{Supp}_x^{unif}(\psi_F\mathbb{C}_X) = \mathrm{Supp}_x(\psi_F\mathbb{C}_X) \times (\mathbb{C}^*)^{r_x},$$

where the coordinates of the last term $(\mathbb{C}^*)^{r_x}$ correspond to t_j such that $f_j(x) \neq 0$. Hence (abusing notation a bit)

$$\begin{aligned}\mathrm{Supp}_x^{unif}(\psi_F\mathbb{C}_X) \cap (S')^* &= \bigcup_{x\in D'} [\mathrm{Supp}_x(\psi_F\mathbb{C}_X) \cap (S')^*] \times [(\mathbb{C}^*)^{r_x} \cap (S')^*] \\ &= \bigcup_{x\in D'} [\mathrm{Supp}_x(\psi_F\mathbb{C}_X \overset{L}{\underset{A}{\otimes}} A')] \times [(\mathbb{C}^*)^{r_x} \cap (S')^*] \subset (S')^*\end{aligned}$$

The claim follows if we show that the last formula is equal to

$$\bigcup_{x\in D'} \mathrm{Supp}_x^{unif}(\psi_G\mathbb{C}_X) = \bigcup_{x\in D'} \mathrm{Supp}_x(\psi_G\mathbb{C}_X) \times (\mathbb{C}^*)^{r'_x} \subset (S')^*,$$

where the coordinates of the last term $(\mathbb{C}^*)^{r'_x}$ correspond to $u_{j'}$ such that $g_{j'}(x) \neq 0$.

By [32, 2.3.8], since ϕ is surjective,

$$\mathrm{Supp}_x(\psi_F\mathbb{C}_X \overset{L}{\underset{A}{\otimes}} A') = \mathrm{Supp}_x(\psi_G\mathbb{C}_X) \subset (S')^*.$$

For the remaining claim, write

$$f'_{j'} = [(\tau_1, \dots, \tau_r) \mapsto \tau_1^{m_{j',1}} \dots \tau_r^{m_{j',r}}].$$

It is not difficult to see then that

$$\phi = [t_j \mapsto \prod_{j'} u_{j'}^{m_{j',j}}].$$

Note that the coordinates on $(\mathbb{C}^*)^{r_x} \cap (S')^*$ correspond to those $j' \in \{1, \dots, r'\}$ such that $u_{j'}$ does not appear in $\phi(t_j)$ for any j with $f_j(x) = 0$. These are the same as the elements of the set

$$\{j' \in \{1, \dots, r'\} \mid m_{j',j} = 0 \text{ for all } j \text{ with } f_j(x) = 0\}.$$

However, we see from the description of F' that this set is the same as

$$\{j' \in \{1, \dots, r'\} \mid g_{j'}(x) \neq 0\}.$$

Hence

$$(\mathbb{C}^*)^{r_x} \cap (S')^* = (\mathbb{C}^*)^{r'_x},$$

as claimed. ∎

Example 3.31. $F = (f_1, \dots, f_r)$ specializes to $G = \prod_{j=1}^r f_j$. To see this, in Definition 3.29 let $F' : S \to S' = \mathbb{C}$ be the map

$$(\tau_1, \dots, \tau_r) \mapsto \tau_1 \cdot \ldots \cdot \tau_r.$$

Here $(S')^* \to S^*$ is the diagonal inclusion and $\phi : A \to A' = \mathbb{C}[u, u^{-1}]$ is given by

$$t_j \mapsto u.$$

In this case, the uniform support of $\psi_G \mathbb{C}_X$ at x is the same as the usual support, i.e.

$$Supp_x^{unif} \psi_G \mathbb{C}_X = Supp_x \psi_G \mathbb{C}_X,$$

and it consists of the eigenvalues of the monodromy on the Milnor fiber of G at x by Remark 3.3.

Example 3.32. $F = (f_1, \dots, f_r)$ specializes to $G = (f_1, \dots, f_{r-1})$. To see this, in Definition 3.29 let $F' : S \to S' = \mathbb{C}^{r-1}$ be defined by

$$(\tau_1, \dots, \tau_r) \mapsto (\tau_1, \dots, \tau_{r-1}).$$

Here the inclusion $(S')^* \to S^*$ is given by

$$(\tau_1, \dots, \tau_{r-1}) \mapsto (\tau_1, \dots, \tau_{r-1}, 1),$$

and the map $A \to A'$ is the natural one induced by the quotient $A' = A/(t_r - 1)$.

3.33. **Thom-Sebastiani.** Next, we state a multiplicative Thom-Sebastiani type of result for the support of the Sabbah specialization complex. First, we have:

Lemma 3.34. *For $i = 1, 2$, let $X_i = \mathbb{C}^{n_i}$, $F_i = (f_{i1}, \dots, f_{ir_i})$, $D_i = \cup_j f_{ij}^{-1}(0)$, $x_i \in D_i$. Then, in $(\mathbb{C}^*)^{r_1} \times (\mathbb{C}^*)^{r_2}$, we have*

$$Supp_{(x_1,x_2)}^{unif} \psi_{F_1 \times F_2} \mathbb{C}_{X_1 \times X_2} = Supp_{x_1}^{unif} \psi_{F_1} \mathbb{C}_{X_1} \times Supp_{x_2}^{unif} \psi_{F_2} \mathbb{C}_{X_2}.$$

Proof. Using the notation of Lemma 3.4 adapted to our situation, there is an equality of local systems of $A_1 \otimes A_2$-modules

$$\mathcal{L}^{F_1 \times F_2} = \mathcal{L}^{F_1} \boxtimes \mathcal{L}^{F_2},$$

where $\boxtimes$ denotes the external direct product on $U_1 \times U_2$, with $U_i = X_i - D_i$, and A_i is the coordinate ring of $S_i^* = (\mathbb{C}^*)^{r_i}$. Using then Lemma 3.4 and standard arguments, one can show that

$$\psi_{F_1 \times F_2} \mathbb{C}_{X_1 \times X_2} = \psi_{F_1} \mathbb{C}_{X_1} \overset{L}{\boxtimes} \psi_{F_2} \mathbb{C}_{X_2},$$

Hence the claim holds for the usual supports. See also [29, Proposition 3.1] for the same statement for the characteristic varieties. The claim for the uniform support follows easily from Definition 3.20. $\square$

The following is the multiplicative Thom-Sebastiani property for the support of the Sabbah specialization complex:

Proposition 3.35. *With notation as in Lemma 3.34, let $r = r_1 = r_2$. Let G be the map*

$$G = F_1 \cdot F_2 : X_1 \times X_2 \longrightarrow (\mathbb{C}^*)^r$$

defined by

$$(x_1, x_2) \mapsto (f_{11}(x_1)f_{21}(x_2), \dots, f_{1r}(x_1)f_{2r}(x_2))$$

for $x_i \in X_i$, $i = 1, 2$. Then

$$Supp^{unif}_{(x_1,x_2)}\psi_G\mathbb{C}_{X_1\times X_2} = \bigcap_{i=1,2} Supp^{unif}_{x_i}\psi_{F_i}\mathbb{C}_{X_i}.$$

Proof. Let $S = \mathbb{C}^r$ and let $F' : S \times S \to S$ be defined by multiplication coordinatewise. Then $G = F' \circ (F_1 \times F_2)$ and thus $F_1 \times F_2$ specializes to G, cf. Definition 3.29. Hence, by Proposition 3.30, $\mathrm{Supp}^{unif}_{(x_1,x_2)}\psi_{F_1\times F_2}\mathbb{C}_{X_1\times X_2}$ specializes to $\mathrm{Supp}^{unif}_{(x_1,x_2)}\psi_G\mathbb{C}_{X_1\times X_2}$ via intersection with the diagonal in $S^* \times S^*$. The claim then follows from Lemma 3.34. □

4. Proofs

4.1. **Proof of Theorem 1.** The method of the proof is to use specialization of polynomial maps to reduce the statement to the case $r = 1$ which is already known by the theorem of Malgrange and Kashiwara. The setup and notation are as in 3.1.

Let $m \in \mathbb{N}^r - \{(0, \dots, 0)\}$. Then, conform Definition 3.29, the polynomial map

$$F = (f_1, \dots, f_r) : X \to S$$

specializes to the polynomial

$$G_m = f_1^{m_1} \dots f_r^{m_r} : X \longrightarrow \mathbb{C}.$$

The induced inclusion $\mathbb{C}^* \hookrightarrow S^*$ is the given by $\lambda \mapsto (\lambda^{m_1}, \dots, \lambda^{m_r})$. Let C_m be the image in S^* of this inclusion map. Note that S^* is the closure of the union of the curves C_m

$$S^* = \left[\bigcup_{m\in\mathbb{N}^r-\{(0,\dots,0)\}} C_m \right]^{cl}.$$

Since $B_{F,x}$ is generated by polynomials with rational coefficients by assumption, the algebraic set $Exp(V(B_{F,x}))$ is defined over $\mathbb{Q}$. Thus

$$Exp(V(B_{F,x})) = \left[\bigcup_{m\in\mathbb{N}^r-\{(0,\dots,0)\}} Exp(V(B_{F,x})) \cap C_m \right]^{cl}$$

By definition, $B_{F,x}$ consists of polynomials $b(s_1, \dots, s_r)$ such that

$$b(s_1, \dots, s_r) \prod_{j=1}^{r} f_j^{s_j} = P \prod_{j=1}^{r} f_j^{s_j+1}$$

for some algebraic differential operator $P(s_1, \ldots, s_r)$. Thus, specializing this equality,

$$Exp(V(B_{F,x})) \cap C_m \supset Exp(V(B_{G_m,x})),$$

where $B_{G_m,x}$ is the classical one-variable local Bernstein-Sato polynomial of G_m. Thus, by Malgrange and Kashiwara,

$$Exp(V(B_{F,x})) \supset \left[\bigcup_{m\in\mathbb{N}^r-\{(0,\ldots,0)\}} \bigcup_{y\in G_m^{-1}(0), y\in D \text{ near } x} \mathrm{Supp}_y^{unif} \psi_{G_m}\mathbb{C}_X \right]^{cl}.$$

By Proposition 3.30, the set on the right-hand side is equal to

$$\left[\bigcup_{m\in\mathbb{N}^r-\{(0,\ldots,0)\}} \bigcup_{y\in G_m^{-1}(0), y\in D \text{ near } x} \mathrm{Supp}_y^{unif} \psi_F\mathbb{C}_X \cap C_m \right]^{cl} =$$

$$= \bigcup_{y\in D \text{ near } x} \mathrm{Supp}_y^{unif} \psi_F\mathbb{C}_X,$$

with the last equality being true since the supports are varieties defined over $\mathbb{Q}$. $\square$

4.2. **Proof of Corollary 1.** It follows from Theorem 1 and Proposition 3.25.

4.3. **Proof of Theorem 8.** Recall that in the situation of Definition 3.29, the pull-back of local systems defines an inclusion

$$L((S')^*) = (S')^* \hookrightarrow L(S^*) = S^*.$$

There is a corresponding inclusion of tangent spaces at identity

$$S' \hookrightarrow S$$

recovering the previous inclusion via $Exp(\cdot)$.

We have an equality of multi-variable topological zeta functions:

$$Z_G^{top} = Z_{F'\circ F}^{top} = (Z_F^{top})_{|S'}.$$

Thus, for polar loci,

$$PL(Z_G^{top}) = PL((Z_F^{top})_{|S'}) \subset PL(Z_F^{top}) \cap S'.$$

By assumption,

$$Exp(PL(Z_F^{top}) \cap S') \subset \bigcup_{x\in D'} \mathrm{Supp}_x^{unif}(\psi_F\mathbb{C}_X) \cap (S')^*,$$

where D' is the union of the zero loci of the $g_{j'}$. By Proposition 3.30, the right-hand side equals

$$\bigcup_{x\in D'} \mathrm{Supp}_x^{unif}(\psi_G\mathbb{C}_X).$$

Hence the Topological Multi-Variable Monodromy Conjecture, that is, Conjecture 5, holds for G. $\square$

5. Hyperplane arrangements

5.1. **Terminology.** Let $D = \cup_{j=1}^{r} D_j$ be a finite collection of hyperplanes in $X = \mathbb{C}^n$. Let f_j be a linear polynomial defining D_j. Let $f = \prod_j f_j^{m_j}$ with $m_j \geq 1$ be a possibly non-reduced polynomial such that the zero locus $V(f)$ is D. We call both D and f *hyperplane arrangements*. The hyperplane arrangement is called *central* if each D_j is a linear subspace of codimension one of X, that is, if f is homogeneous. A central hyperplane arrangement f is *indecomposable* if there is no linear change of coordinates on X such that f can be written as the product of two non-constant polynomials in disjoint sets of variables. Note that indecomposability is a property of the underlying reduced zero locus D of f. An *edge* is an intersection of hyperplanes D_j. An arrangement is *essential* if $\{0\}$ is an edge. For a linear subset $W \subset X$, the *restriction of D to W* is the hyperplane arrangement D_W given by the image of $\cup_{D_j \supset W} D_j$ in the vector space quotient X/W, which is defined as soon as we make a linear change of coordinates such that $0 \in W$. Similarly, one defines *the restriction f_W of f to W* as a polynomial map on X/W, by keeping track of the multiplicities along the hyperplanes D_j which contain W. An edge W of D is called *dense* if the restriction arrangement D_W is indecomposable. For example, D_j is a dense edge for every j. The *canonical log resolution* of D is the map $\mu : Y \to \mathbb{C}^n$ obtained by composition of, in increasing order for $i = 0, 1, \ldots, n-2$, the blowups along the (proper transform of) the union of the dense edges of dimension i.

Theorem 5.2. *(* [33, Theorem 3.1]*) The canonical log resolution $\mu : Y \to X$ is a log resolution of f.*

Proposition 5.3. *(*[33, Proposition 2.6]*) If f is a central hyperplane arrangement and $V = \mathbb{P}^{n-1} - \mathbb{P}(D)$, then f is indecomposable if and only if $\chi(V) \neq 0$.*

Remark 5.4. The Euler characteristic of the complement of a hyperplane arrangement can be determined only from the lattice of intersections of the hyperplanes in the arrangement, see [28]. Hence the previous Proposition also implies that indecomposability and density are combinatorial conditions.

5.5. **Sabbah specialization complex for arrangements.** From now on we use the same setup as in 3.1. Assume that f_j are central hyperplane arrangements in $X = \mathbb{C}^n$, not necessarily reduced, of degree d_j.

The following two lemmas are immediate consequences of Corollary 3.9, Proposition 3.25, and Proposition 5.3:

Lemma 5.6. *If $f = \prod_{j=1}^{r} f_j$ is an indecomposable central hyperplane arrangement, then $V(t_1^{d_1} \cdots t_r^{d_r} - 1) \subset Supp_0(\psi_F \mathbb{C}_X)$.*

Lemma 5.7. *There is an equality in Lemma 5.6, if, in addition, f is reduced.*

Definition 5.8. *We say that polynomial map $F = (f_1, \ldots, f_r)$ with $f_j \in \mathbb{C}[x_1, \ldots, x_n]$ splits into $G \cdot H$, and that $G \cdot H$ is a splitting of F, if, up to a different choice of coordinates, there exists m with $1 \leq m \leq n$ and there are polynomials $g_j(x_1, \ldots, x_m)$ and $h_j(x_{m+1}, \ldots, x_n)$ for $j = 1, \ldots, r$, such that not all g_j are constant, not all h_j*

are constant, and $f_j = g_j h_j$*. If so, we set* $G = (g_1, \dots, g_r)$ *and* $H = (h_1, \dots, h_r)$*. Otherwise, we say that* F *is does not split. We say that a splitting*

$$F = F^{(1)} \cdot \ldots \cdot F^{(l)}$$

is total if each $F^{(i)}$ *does not split.*

Let $F = (f_1, \dots, f_r)$ be such that f_j are linear forms defining mutually distinct hyperplanes. Up to multiplication by constants, a total splitting of F is unique. For an edge W of the hyperplane arrangement $f = \prod_{j=1}^r f_j$, let

$$F_W = (f_{1,W}, \dots, f_{r,W}) : X/W \longrightarrow S^*,$$

where $f_{j,W}$ is the restriction of the hyperplane arrangement f_j to W as defined in 5.1. More precisely, $f_{j,W} = f_{j|X/W}$ if $f_j(W) = 0$, and $f_{j,W} = 1$ otherwise. Note that W is a dense edge if and only if F_W does not split. For every edge, let

$$F_W = \prod_{i=1}^{l_W} F_W^{(i)}$$

be a total splitting of F_W. If we set $F_W^{(i)} = (f_{1,W}^{(i)}, \dots, f_{r,W}^{(i)})$, let

$$d_{j,W}^{(i)} := \deg f_{j,W}^{(i)}.$$

Note that $d_{j,W}^{(i)}$ is either 0 or 1.

Proposition 5.9. *(a) If* f_j *are linear forms defining mutually distinct hyperplanes, then*

$$\bigcup_{x \in D} Supp_x^{unif}(\psi_F \mathbb{C}_X) = \bigcup_W V((t_1^{d_{1,W}^{(i)}} \dots t_r^{d_{r,W}^{(i)}} - 1 \mid i = 1, \dots, l_W)), \tag{3}$$

where the union is over the edges W *of the hyperplane arrangement* $\prod_{j=1}^r f_j$*. In particular, the codimension-one part is the zero locus in* S^* *of*

$$\prod_W \left(\prod_{j \,:\, f_j(W)=0} t_j - 1 \right),$$

where the first product is over dense edges W *of* $f = \prod_j f_j$*.*

(b) Assuming Conjecture 2, formula (3) also holds for $Exp(V(B_F))$*.*

Proof. First, we assume that $f = \prod_{j=1}^r f_j$ is a central hyperplane arrangement. Then

$$\bigcup_{x \in D} \mathrm{Supp}_x^{unif}(\psi_F \mathbb{C}_X) = \mathrm{Supp}_0^{unif}(\psi_F \mathbb{C}_X) \cup \bigcup_{0 \neq y \in D} \mathrm{Supp}_y^{unif}(\psi_F \mathbb{C}_X).$$

Let us focus on the first term of the right-hand side. Let

$$F_0 = F = F_0^{(1)} \cdot \ldots \cdot F_0^{(l_0)}$$

be a total splitting of F_0, with $F_0^{(i)}$ defined on X_i, and $X = \times_{i=1}^{l_0} X_i$. By Proposition 3.35, we have

$$\mathrm{Supp}_0^{unif}(\psi_F \mathbb{C}_X) = \bigcap_{i=1}^{l_0} \mathrm{Supp}_0^{unif}(\psi_{F_0^{(i)}} \mathbb{C}_{X_i}).$$

Since $F_0^{(i)}$ is does not split, the hyperplane arrangement $\prod_{j=1}^r f_{j,0}^{(i)}$ is indecomposable. Hence, by Proposition 5.3, for each $F_0^{(i)}$ we are in the case Lemma 5.7. Thus,

$$\mathrm{Supp}_0(\psi_{F_0^{(i)}} \mathbb{C}_{X_i}) = V(t_1^{d_{1,0}^{(1)}} \ldots t_r^{d_{r,0}^{(l_0)}} - 1)$$

inside the torus

$$\mathrm{Spec}\,(\mathbb{C}[t_1^{\pm d_{1,0}^{(1)}}, \ldots, t_r^{\pm d_{r,0}^{(l_0)}}]).$$

By the definition of the uniform support, $\mathrm{Supp}_0(\psi_{F_0^{(i)}} \mathbb{C}_{X_i})$ has the same equations, but inside the possibly-bigger torus

$$\mathrm{Spec}\,(\mathbb{C}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]) = S^*.$$

Thus,

$$\begin{aligned}\mathrm{Supp}_0^{unif}(\psi_F \mathbb{C}_X) &= \bigcap_{i=1}^{l_0} V(t_1^{d_{1,0}^{(1)}} \ldots t_r^{d_{r,0}^{(l_0)}} - 1) \\ &= V(t_1^{d_{1,0}^{(1)}} \ldots t_r^{d_{r,0}^{(l_0)}} - 1 \mid i = 1, \ldots, l_0).\end{aligned}$$

The rest of the claim follows by replacing $W = 0$ in the above argument with other edges W of the hyperplane arrangement $\prod_{j=1}^r f_j$.

If f is not central, fix $x \in D$. The above argument for the central case gives the equations of the support of $\psi_F \mathbb{C}_X$ at x inside the torus

$$\mathbb{T}_x = \bigcup_{j: f_j(x) \neq 0} V(t_j - 1) \subset S^*.$$

By the definition of the uniform support, $\mathrm{Supp}_x^{unif}(\psi_F \mathbb{C}_X)$ has the same equations in S^*, and the claim follows. $\square$

5.10. **Proof of Corollary 3.** First, we need to clarify the notation used in the statement. With the notation as in Proposition 5.9, for an edge W of D let f_W be the restriction of the hyperplane arrangement f to W as in 5.1. If

$$f_W^{(i)} := \prod_{j=1}^r f_{j,W}^{(i)},$$

then

$$f_W = \prod_{i=1}^{l_W} f_W^{(i)}$$

is a total splitting of f_W. Let

$$d_W^{(i)} := \deg f_W^{(i)} = \sum_{j=1}^{r} d_{j,W}^{(i)}.$$

Let f_j for $j = 1, \ldots, r$, be the irreducible factors of f. Specialize $F = (f_1, \ldots, f_r)$ to $f = \prod_{j=1}^{r} f_j$ as in Example 3.31. As in the proof of Theorem 8,

$$\bigcup_{x \in D} \mathrm{Supp}_x^{unif}(\psi_F \mathbb{C}_X)$$

specializes via the restriction to the diagonal $t_1 = \ldots = t_r$ to the set

$$\star = \bigcup_{x \in D} \bigcup_{k} \{\text{eigenvalues of monodromy on } H^k(M_{f,x}, \mathbb{C})\},$$

where $M_{f,x}$ is the Milnor fiber of f at x, see Remark 3.3. Since

$$d_W^{(i)} = \sum_{j=1}^{r} d_{j,W}^{(i)},$$

we have

$$V(t_1 = \ldots = t_r = t) \cap \bigcup_{x \in D} \mathrm{Supp}_x^{unif}(\psi_F \mathbb{C}_X) = \bigcup_{W} V(t^{d_W^{(i)}} - 1 \mid i = 1, \ldots, l_W),$$

by Proposition 5.9. The conclusion follows from the fact that $Exp(V(b_f)) = \star$, by the theorem of Malgrange and Kashiwara. $\square$

5.11. **Proof of Theorem 5.** We deal first with the central case. For $j = 1, \ldots, r$, let f_j be central hyperplane arrangements in $X = \mathbb{C}^n$. Using the canonical log resolution, we see that the polar locus $PL(Z_F^{top})$ is a hyperplane sub-arrangement of $\cup_W P_W$, with

$$P_W = \{a_{W,1}s_1 + \ldots + a_{W,r}s_r + k_W + 1 = 0\},$$

where W varies over the dense edges of the hyperplane arrangement D, $a_{W,j} = \mathrm{ord}_W(f_j)$, and $k_W = \mathrm{codim}\,(W) - 1$.

Fix a dense edge W, and the corresponding hyperplane P_W which candidates for a component of the polar locus. Let D_W, f_W, $f_{j,W}$ be the restrictions of the hyperplane arrangements D, f, f_j, respectively, to W as defined in 5.1. We have $f_W = \prod_j f_{j,W}$, where the product is over those j with $f_j(W) = 0$. We can assume $\{j \mid f_j(W) = 0\} = \{1, \ldots, r'\}$ for some integer r'. Now, take a point $x \in W$ not lying on any hyperplane in D which does not contain W. After choosing a splitting of $W \subset \mathbb{C}^n$, we have locally around x, $D = D_W \times W \subset \mathbb{C}^n = \mathbb{C}^n/W \times W$ and $f = f_W \cdot u$, where u is a (locally) invertible function. Hence, by Lemma 5.6,

$$V(\prod_{j=1}^{r'} t_j^{a_{W,j}} - 1) \subset \mathrm{Supp}_x(\psi_{f_{W,1},\ldots,f_{W,r'}} \mathbb{C}_{\mathbb{C}^n/W}) \subset (\mathbb{C}^*)^{r'}.$$

On the other hand, by the definition of the uniform support, we have

$$V(\prod_{j=1}^{r'} t_j^{a_{W,j}} - 1) \subset \mathrm{Supp}_x^{unif}(\psi_F \mathbb{C}_X) \subset (\mathbb{C}^*)^{r'} \times (\mathbb{C}^*)^{r-r'} = S^*.$$

Let $\lambda \in P_W$. Then

$$\prod_{j=1}^{r} \left(e^{2\pi i \lambda_j}\right)^{a_{W,j}} - 1 = e^{-2\pi i \cdot codim(W)} - 1 = 0.$$

This shows that

$$Exp(P_W) \subset V(t_1^{a_{W,1}} \cdots t_r^{a_{W,r}} - 1) \subset \mathrm{Supp}_x^{unif}(\psi_{f_1,\ldots,f_r}\mathbb{C}_X),$$

which was the claim.

The non-central case follows as in the proof of Proposition 5.9. □

5.12. **Proof of Theorem 6.** We prove the claim for the case when $f = \prod_{j=1}^r f_j$ is central, since this imples the non-central case as well. As in the proof of Theorem 5, the polar locus $PL(Z_F^{top})$ is included in the hyperplane arrangement $\cup_W P_W$, with

$$P_W = \{a_{W,1}s_1 + \ldots + a_{W,r'}s_{r'} + k_W + 1 = 0\},$$

where W varies over the dense edges of the hyperplane arrangement D, $a_{W,j} = \mathrm{ord}_W(f_j)$, and $k_W = \mathrm{codim}\,(W) - 1$. Hence,

$$P_W = \{\deg(f_{1,W})s_1 + \ldots \deg(f_{r',W})s_{r'} + \dim(\mathbb{C}^n/W) = 0\}.$$

Note that $f_W = \prod_{j=1}^{r'} f_{W,j}$ is indecomposable, and automatically central and essential. By assumption, P_W is in the zero locus of the ideal B_{F_W}, where $F_W = (f_{1,W}, \ldots, f_{r',W})$ as before.

Now, as in the proof of Theorem 5, take a point $x \in W$ not lying on any hyperplane in D which does not contain W. We have $B_{F_W} = B_{F,x}$, and the zero locus of $B_{F,x}$ is included in the zero locus of B_F. The claim follows. □

6. $\mathcal{D}$-modules

6.1. **Bernstein-Sato ideals and $\mathcal{D}$-modules.** We recall the $\mathcal{D}$-module theoretic interpretation of Bernstein-Sato ideals. We use it to describe the zero locus $V(B_F)$ in terms of zero loci of other ideals of Bernstein-Sato type, but which can be more amenable to computations. There are different ways to define ideals of Bernstein-Sato type, so we start with a more general definition.

Let $X = \mathbb{C}^n$ and let D_X denote the Weyl algebra of algebraic differential operators on X,

$$D_X = \mathbb{C}\left[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}\right]$$

with the usual commuting relations.

Let $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise. We will denote by $\mathbf{1}$ the vector $(1, \ldots, 1)$ in $\mathbb{Z}^r$, and by $\mathbf{1}^{(j)}$ the vector in $\mathbb{Z}^r$ with i-th coordinate $\delta_{i,j}$.

Let $F = (f_1, \dots, f_r)$ with $f_j \in \mathbb{C}[x_1, \dots, x_n]$. Let

$$\mathcal{U} = \{u^{(k)} \in \mathbb{N}^r \mid k = 1, \dots, l\}$$

be a collection of vectors.

Definition 6.2. *The Bernstein-Sato ideal associated to F and $\mathcal{U}$ is the ideal*

$$B_{\mathcal{U},F} \subset \mathbb{C}[s_1, \dots, s_r]$$

of all polynomials $b(s_1, \dots, s_r)$ such that

$$b(s_1, \dots, s_r) \prod_{j=1}^{r} f_j^{s_j} = \sum_{k=1}^{l} P_k \prod_{j=1}^{r} f_j^{s_j + u_j^{(k)}}$$

for some algebraic differential operators P_k in $D_X[s_1, \dots, s_r]$. When $\mathcal{U}$ consists of only one element u, we denote $B_{\mathcal{U},F}$ by $B_{u,F}$.

Remark 6.3. (1) B_F, as defined before, is $B_{\mathbf{1},F}$.

(2) For a point x in X, there are local versions $B_{\mathcal{U},F,x}$ of the Bernstein-Sato ideals defined using the germ of F at x, see also the next subsection. The existence of non-zero such ideals $B_{\mathcal{U},F,x}$ has been proved by Sabbah [31], see also [3], [17].

One has the following correspondence, see [17, Lemma 1.5]:

Lemma 6.4. *By the correspondence $s_j \leftrightarrow -\partial_{t_j} t_j$ and $f_j \leftrightarrow \delta(t_j - f_j)$, where $\delta(u)$ denotes the Dirac delta function, i.e. the standard generator of $D_{\mathbb{A}^1}/D_{\mathbb{A}^1}u$ with u the affine coordinate on $\mathbb{A}^1$, there is an isomorphism of $D_X[s_1, \dots, s_r]$-modules*

$$D_X[s_1, \dots, s_r] \prod_{j=1}^{r} f_j^{s_r} \cong D_X[-\partial_{t_1} t_1, \dots, -\partial_{t_r} t_r] \prod_{j=1}^{r} \delta(t_j - f_j).$$

The action of t_j on the right-hand side corresponds to replacing s_j by $s_j + 1$ on left-hand side.

Let $Y = X \times \mathbb{C}^r$ with affine coordinates $x_1, \dots, x_n, t_1, \dots, t_r$. Define for $u \in \mathbb{N}^r$

$$V^u D_Y := D_X \otimes_{\mathbb{C}} \sum_{\substack{\beta,\gamma \in \mathbb{N}^r \\ \beta - \gamma \geq u}} \mathbb{C} t_1^{\beta_1} \dots t_r^{\beta_r} \partial_{t_1}^{\gamma_1} \dots \partial_{t_r}^{\gamma_r} \quad \subset \quad D_Y.$$

The following is the $\mathcal{D}$-module theoretic interpretation of Bernstein-Sato ideals and it is a consequence of Lemma 6.4:

Proposition 6.5. *Let $u \in \mathbb{N}^r$. The Bernstein-Sato ideal $B_{u,F}$ consists of the polynomials $b(s_1, \dots, s_r)$ such that*

$$b(-\partial_{t_1} t_1, \dots, -\partial_{t_r} t_r) \cdot V^0 D_Y \cdot \prod_{j=1}^{r} \delta(t_j - f_j) \subset V^u D_Y \cdot \prod_{j=1}^{r} \delta(t_j - f_j).$$

Proposition 6.6. *Let* $\mathcal{U} \subset \{\mathbf{1}^{(1)}, \dots, \mathbf{1}^{(r)}\}$ *and* $k \in \{1, \dots, r\}$*. We have:*
(a) $B_F \subset B_{\mathbf{1}^{(k)},F}$;
(b) $V(B_F) \supset \bigcup_{j=1}^{r} V(B_{\mathbf{1}^{(j)},F})$;
(c) $Exp(V(B_F)) = \bigcup_{j=1}^{r} Exp(V(B_{\mathbf{1}^{(j)},F}))$;
(d) $B_{\mathcal{U},F} \supset \sum_{\mathbf{1}^{(j)}\in\mathcal{U}} B_{\mathbf{1}^{(j)},F}$;
(e) $V(B_{\mathcal{U},F}) \subset \bigcap_{\mathbf{1}^{(j)}\in\mathcal{U}} V(B_{\mathbf{1}^{(j)},F})$.

Proof. Let $b \in B_F$. Then, for some $P \in D_X[s_1, \dots, s_r]$,

$$b(s)\prod_{j=1}^{r} f_j^{s_j} = P\prod_{j=1}^{r} f_j^{s_j+1} = (P\prod_{j'\neq k} f_{j'})\prod_{j=1}^{r} f_j^{s_j+\delta_{j,k}}.$$

Hence, $b \in B_{\mathbf{1}^{(k)},F}$, and (a) follows. (b) follows from (a).

For (c), we assume for simplicity that $r = 2$. The general case is similar. Let $b_j(s_1, s_2)$ be in $B_{\mathbf{1}^{(j)},F}$. Then

$$b_1(s_1, s_2)f_1^{s_1}f_2^{s_2} = P_1(s_1, s_2)\cdot f_1^{s_1+1}f_2^{s_2}$$
$$b_2(s_1, s_2)f_1^{s_1}f_2^{s_2} = P_2(s_1, s_2)\cdot f_1^{s_1}f_2^{s_2+1},$$

for some $P_j \in \mathcal{D}_X[s_1, s_2]$. Multiplying the first relation by f_2, we have

$$b_1(s_1, s_2)f_1^{s_1}f_2^{s_2+1} = f_2P_1(s_1, s_2)\cdot f_1^{s_1+1}f_2^{s_2}.$$

By Lemma 6.4, the right-hand side is

$$t_2P_1(-\partial_{t_1}t_1, -\partial_{t_2}t_2)\cdot t_1\delta(t_1 - f_1)\delta(t_2 - f_2).$$

This is in turn equal to

$$Q_1(-\partial_{t_1}t_1, -\partial_{t_2}t_2)\cdot t_2t_1\delta(t_1 - f_1)\delta(t_2 - f_2),$$

for some $Q_1 \in \mathcal{D}_X[s_1, s_2]$ due to repeated shifts $\partial_{t_2}t_2 = t_2\partial_{t_2} + 1$. Thus, by Lemma 6.4 again,

$$b_1(s_1, s_2)f_1^{s_1}f_2^{s_2+1} = Q_1(s_1, s_2)\cdot f_1^{s_1+1}f_2^{s_2+1}.$$

Applying P_2 on both sides, we get

$$\widetilde{b}_1(s_1, s_2)b_2(s_1, s_2)f_1^{s_1}f_2^{s_2} = P_2Q_1f_1^{s_1+1}f_2^{s_2+1},$$

where $\widetilde{b}_1$ is obtained from b_1 by some shifts of s_1 and s_2 by integers. Hence $\widetilde{b}_1b_2$ is in B_F, and

$$Exp(V(B_{\mathbf{1}^{(1)},F})) \cup Exp(V(B_{\mathbf{1}^{(2)},F})) \supset Exp(V(B_F)).$$

The inverse inclusion, and hence equality, follows (b). This proves (c).

(d) follows by definition, (e) follows from (d). □

The above Proposition also holds in the local case, for Bernstein-Sato ideals at a point x.

6.7. **The remainder of Conjecture 2.** It is unlikely that geometric methods can be pushed to prove the converse of Theorem 1, and thus to complete the proof of Conjecture 2. While this can be phrased in terms of $\mathcal{D}$-modules using Sabbah specialization for $\mathcal{D}$-modules, see [32, §4 and §5], we opt for a different strategy which has the advantage of being easier to state. To be more precise, we will prove Theorem 4 assuming, instead of Conjecture 2, the Conjecture 3.

We use the notation as in 1.1. Let $\mathcal{D}_X$ be the sheaf of holomorphic differential operators on the complex manifold $X = \mathbb{C}^n$. For $F = (f_1, \dots, f_r)$ with $f_j \in \mathbb{C}[x_1, \dots, x_r]$, let

$$f = \prod_{j=1}^{r} f_j.$$

Consider the following $\mathcal{D}_X$-submodule of $\mathcal{O}_X[f^{-1}, s_1, \dots, s_r]f_1^{s_1} \dots f_r^{s_r}$

$$\mathcal{N} = \mathcal{D}_X[s_1, \dots, s_r]f_1^{s_1} \dots f_r^{s_r}$$

and the injective map

$$\nabla : \mathcal{N} \longrightarrow \mathcal{N}$$

which sends $(s_1, \dots, s_r)$ to $(s_1 + 1, \dots, s_r + 1)$.

For a point $x \in D$, the local Bernstein-Sato ideal of F at x is

$$B_{F,x} = \{b \in \mathbb{C}[s_1, \dots, s_r] \mid b(s_1, \dots, s_r)\mathcal{N}_x \subset \nabla\mathcal{N}_x\}.$$

Conjecture 3 is equivalent to: if $\alpha \in V(B_{F,x})$ then, locally at x,

$$\sum_{j=1}^{r} (s_j - \alpha_j)\mathcal{N}/\nabla\mathcal{N} \subsetneq \mathcal{N}/\nabla\mathcal{N}.$$

Remark 6.8. When $r = 1$, Conjecture 3 is true. In this case, $\mathcal{N}/\nabla\mathcal{N}$ is a holonomic $\mathcal{D}_X$-module, hence artinian. Thus the map $s - \alpha$ on $\mathcal{N}/\nabla\mathcal{N}$ is surjective if and only if it is an isomorphism, or, in other words, if and only if α is not a root of the classical one-variable Bernstein-Sato polynomial of f.

The converse of Theorem 1 boils down, in the case when the polynomials f_j with $f_j(x) = 0$ define mutually distinct reduced and irreducible hypersurface germs at x, to Conjecture 3 by the following:

Proposition 6.9. *Assume that f_j with $f_j(x) = 0$ define mutually distinct reduced and irreducible hypersurface germs at x. If Conjecture 3 is true, then*

$$Exp(V(B_{F,x})) \subset \bigcup_{y \in D \text{ near } x} \mathcal{V}^{unif}(U_{F,y}).$$

Proof. We follow the strategy as in the case $r = 1$ from [5, 6.3.5]. Let $\alpha \in V(B_{F,x})$. The element $Exp(\alpha) \in S^*$ defines an element in $L(U_{F,x})$. We can take $U_{F,y} \subset U_{F,x}$ for y near x. By restriction, $Exp(\alpha)$ also defines an element in $L(U_{F,y})$ for y near x. Let $\mathcal{L}_\alpha$ denote the rank one local system on $U_{F,x}$ determined by $Exp(\alpha)$.

Suppose that $Exp(\alpha) \notin \mathcal{V}^{unif}(U_{F,y})$ for all $y \in D$ near x. Then $H^k(U_{F,y}, \mathcal{L}_\alpha) = 0$ for all k and all $y \in D$ near x. In particular, $j_*\mathcal{L}_\alpha$ has no sections over a small ball $U_y \subset U_x$ in X around y, where $j : U_{F,x} \to U_x$ is the natural open inclusion. Hence

$$(j_*\mathcal{L}_\alpha)_{|U_x \cap D} = 0.$$

In U_x,

$$\mathcal{L}_\alpha = DR_X(\mathcal{M}_{-\alpha}),$$

where DR_X is the de Rham functor and $\mathcal{M}_{-\alpha}$ is the $\mathcal{D}_X$-module

$$\mathcal{M}_{-\alpha} = \mathcal{O}_X[f^{-1}]f_1^{-\alpha_1} \dots f_r^{-\alpha_r}.$$

Because $DR_X(\mathcal{M}_{-\alpha})_{|U_x \cap D} = 0$, the stalks $(\mathcal{M}_{-\alpha})_x$ and $(\mathcal{M}_\alpha)_x$ are simple $\mathcal{D}_{X,x}$-modules, the proof of this claim being the same as for [5, 6.3.14].

We consider

$$\mathcal{D}_X f_1^{\alpha_1} \dots f_r^{\alpha_r}$$

as a $\mathcal{D}_X$-submodule of $\mathcal{M}_\alpha$. We will show that the assumptions imply that, locally at x,

$$\mathcal{D}_X f_1^{\alpha_1+1} \dots f_r^{\alpha_r+1} \subsetneq \mathcal{D}_X f_1^{\alpha_1} \dots f_r^{\alpha_r},$$

contradicting the simplicity of $(\mathcal{M}_\alpha)_x$.

Let

$$\mathcal{N}_\alpha = \mathcal{N} / \sum_{j=1}^{r} (s_j - \alpha_j)\mathcal{N}.$$

Then ∇ induces a map

$$\rho_\alpha : \mathcal{N}_{\alpha+\mathbf{1}} \longrightarrow \mathcal{N}_\alpha$$

Since we are assuming that

$$\sum_{j=1}^{r} (s_j - \alpha_j)\mathcal{N}/\nabla\mathcal{N} \subsetneq \mathcal{N}/\nabla\mathcal{N},$$

the map ρ_α is not surjective.

There is a natural commutative diagram of $\mathcal{D}_X$-modules

$$\begin{tikzcd}
\mathcal{N}_{\alpha+\mathbf{1}} \arrow[r, "\rho_\alpha"] \arrow[d, two heads] & \mathcal{N}_\alpha \arrow[d, two heads] \\
\mathcal{D}_X f_1^{\alpha_1+1} \dots f_r^{\alpha_r+1} \arrow[r, hook] & \mathcal{D}_X f_1^{\alpha_1} \dots f_r^{\alpha_r}
\end{tikzcd}$$

where the vertical maps are surjective and the bottom map is an inclusion.

Replace, if necessary, α by $\alpha - v \cdot \mathbf{1}$ for some positive integer v to obtain that $\alpha \in V(B_{F,x})$, but

$$\alpha - \nu \cdot \mathbf{1} \notin V(B_{F,x}), \quad \text{for all } \nu \in \mathbb{Z}_{>0}.$$

It is possible to do so by [27, Proposition 3.2]. Note that $\mathcal{M}_\alpha$ is unchanged. Then, by the local version of [27, Proposition 3.6], the right-most map gives an isomorphism

$$(\mathcal{N}_\alpha)_x \cong \mathcal{D}_{X,x} f_1^{\alpha_1} \dots f_r^{\alpha_r}.$$

Since ρ_α is not surjective, it follows that

$$\mathcal{D}_{X,x} f_1^{\alpha_1+1} \dots f_r^{\alpha_r+1} \subsetneq \mathcal{D}_{X,x} f_1^{\alpha_1} \dots f_r^{\alpha_r},$$

which is what was claimed. □

7. Example

The following example is just beyond the borderline of what Bernstein-Sato ideals B_F can be computed with currently implemented algorithms and existing hardware.

Example 7.1. Let $F = (x, y, x + y, z, x + y + z)$. Then the product of all entries of F forms a central essential indecomposable hyperplane arrangement in $\mathbb{C}^3$. The Bernstein-Sato ideal B_F of F is currently intractable with both `dmod.lib` in Singular and with `bsi` in risa/asir. However, Conjecture 2 predicts via Corollary 2 that, in $(\mathbb{C}^*)^5$,

$$Exp(V(B_F)) = V((\prod_{j=1}^{5} t_j - 1) \cdot (t_1t_2t_3 - 1)(t_3t_4t_5 - 1)(t_1 \dots t_5 - 1)). \tag{4}$$

We can actually check this as follows. In the notation of Proposition 6.6, one can compute with `dmod.lib` [20]:

$$B_{\mathbf{1}^{(1)},F} = ((s_1 + 1)(s_1 + s_2 + s_3 + 2)(s_1 + s_2 + s_3 + s_4 + s_5 + 3)),$$
$$B_{\mathbf{1}^{(2)},F} = ((s_2 + 1)(s_1 + s_2 + s_3 + 2)(s_1 + s_2 + s_3 + s_4 + s_5 + 3)),$$
$$B_{\mathbf{1}^{(3)},F} = ((s_3 + 1)(s_1 + s_2 + s_3 + 2)(s_3 + s_4 + s_5 + 2)(s_1 + s_2 + s_3 + s_4 + s_5 + 3)),$$
$$B_{\mathbf{1}^{(4)},F} = ((s_4 + 1)(s_3 + s_4 + s_5 + 2)(s_1 + s_2 + s_3 + s_4 + s_5 + 3)),$$
$$B_{\mathbf{1}^{(5)},F} = ((s_5 + 1)(s_3 + s_4 + s_5 + 2)(s_1 + s_2 + s_3 + s_4 + s_5 + 3)).$$

Then (4) follows from part (c) of Proposition 6.6. U. Walther has also checked that (4) holds, using a different computation.

The local topological zeta function $Z_{F,0}^{top}$ of F at the origin has a degree-7 irreducible numerator, and the denominator is equal to

$$\prod_{j=1}^{5}(s_j + 1) \cdot (s_1 + s_2 + s_3 + 2)(s_3 + s_4 + s_5 + 2)(s_1 + \dots + s_5 + 3).$$

This illustrates Theorem 5. Using part (b) of Proposition 6.6, this also shows that Conjecture 4 and the local version of Conjecture 5 hold for F.

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